1 $\displaystyle\int_0^1 \log(x^2+1)\,dx$ 【2025 長崎大】

$$= \Big[x\log(x^2+1)\Big]_0^1 - \int_0^1 x\cdot\frac{2x}{x^2+1}\,dx$$

$$= \log 2 - 2\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx$$

$$= \log 2 - 2\Big(1 - \underbrace{\int_0^1 \frac{1}{x^2+1}\,dx}_{=\frac{\pi}{4}}\Big)$$

$x=\tan\theta$ と置換

$$= \log 2 - 2 + \frac{\pi}{2}.$$

2 $\displaystyle\int_0^1 2x^3\sqrt{x^2+1}\,dx$ 【2025 宮崎大】

($x^2+1=t$ とおく. $2x\,dx=dt$

x	$0\to1$
t	$1\to2$

$x^2=t-1$)

$$= \int_1^2 (t-1)\cdot\sqrt{t}\cdot dt$$

$$= \int_1^2 \left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right)dt$$

$$= \left[\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right]_1^2$$

$$= \frac{2}{5}\cdot4\sqrt{2} - \frac{2}{3}\cdot2\sqrt{2} - \left(\frac{2}{5}-\frac{2}{3}\right)$$

$$= \frac{4}{15}(\sqrt{2}+1).$$

3 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin^3 x}{1-\cos x}\,dx$ 【2025 横浜市立大】

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x\,(1-\cos^2 x)}{1-\cos x}\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,(1+\cos x)\,dx$$

$\cos x = t$ と置換

$$= \int_{\frac{1}{\sqrt{2}}}^{0} (1+t)(-1)\,dt$$

$$= \left[\frac{t^2}{2}+t\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{4}+\frac{1}{\sqrt{2}} = \frac{1+2\sqrt{2}}{4}$$

4 $\displaystyle\int_1^e \frac{(\log x)^2}{x^2}\,dx$ 【2025 山梨大】

$\log x = t$ と置換

$\frac{1}{x}dx = dt$, $x=e^t$

$$= \int_0^1 \frac{t^2}{e^t}\,dt$$

$$= \int_0^1 t^2 e^{-t}\,dt$$

$$= \Big[t^2\cdot(-e^{-t}) - 2t\cdot e^{-t} + 2(-e^{-t})\Big]_0^1$$

$$= \Big[(t^2+2t+2)e^{-t}\Big]_1^0$$

$$= 2 - \frac{5}{e}.$$

1 $\displaystyle\int_1^e x(\log x)^2 dx$ 【2025 弘前大】

$\left(\log x = t \text{ とおく.} \quad x = e^t, \quad dx = e^t dt \right)$

$$= \int_0^1 e^t \cdot t^2 e^t dt = \int_0^1 e^{2t} \cdot t^2 dt$$

$$= \left[\frac{e^{2t}}{2} \cdot t^2 - \frac{e^{2t}}{4} \cdot 2t + \frac{e^{2t}}{8} \cdot 2 \right]_0^1$$

$$= \left[\frac{2t^2 - 2t + 1}{4} e^{2t} \right]_0^1$$

$$= \frac{e^2 - 1}{4}.$$

2 $\displaystyle\int_0^1 \frac{x}{\sqrt{x+1}} dx$ 【2025 群馬大】

$\left(\sqrt{x+1} = t \text{ とおく.} \quad x = t^2 - 1, \quad dx = 2t\,dt \right)$

$$= \int_1^{\sqrt{2}} \frac{t^2 - 1}{t} \cdot 2t\,dt$$

$$= \int_1^{\sqrt{2}} 2(t^2 - 1)\,dt$$

$$= 2\left[\frac{1}{3}t^3 - t \right]_1^{\sqrt{2}}$$

$$= \frac{4 - 2\sqrt{2}}{3}.$$

3 $\displaystyle\int_0^{\frac{\pi}{12}} \sin^2 3x\,dx$ 【2025 福島大】

$$= \int_0^{\frac{\pi}{12}} \frac{1 - \cos 6x}{2} dx$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x \right]_0^{\frac{\pi}{12}}$$

$$= \frac{1}{2}\left(\frac{\pi}{12} - \frac{1}{6} \right) = \frac{\pi - 2}{24}.$$

4 $\displaystyle\int_0^{\frac{\pi}{6}} x \sin 2x\,dx$ 【2025 愛媛大】

ビ　ヤ

$$= \left[x \cdot \frac{\cos 2x}{-2} - 1 \cdot \frac{\sin 2x}{-4} \right]_0^{\frac{\pi}{6}}$$

$$= -\frac{\pi}{12}\cos\frac{\pi}{3} + \frac{1}{4}\sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{8} - \frac{\pi}{24} = \frac{3\sqrt{3} - \pi}{24}.$$

$\boxed{1}$ $\displaystyle\int_{-1}^{1}(x+1)e^x dx$ 【2024 北海道大】

$$= \left[xe^x\right]_{-1}^{1}$$

$$(xe^x)' = 1\cdot e^x + xe^x = (x+1)e^x.$$

$$= e - (-e^{-1})$$

$$= \underline{e + e^{-1}}$$

$$\left(= e + \frac{1}{e}.\right)$$

$\boxed{2}$ $\displaystyle\int_0^1 \frac{x^3}{\sqrt{2-x^2}}dx \quad (=: I)$ 【2025 群馬大】

$\sqrt{2-x^2} = t$ とおく. $2-x^2 = t^2$, $x^2 = 2-t^2$

x	$0 \to 1$
t	$\sqrt{2} \to 1$

$-2x\,dx = 2t\,dt$

$x\,dx = -t\,dt$

$$I = \int_{\sqrt{2}}^{1} \frac{(2-t^2)}{t}(-t)\,dt$$

$$= \int_{\sqrt{2}}^{1}(t^2-2)\,dt$$

$$= \left[\frac{t^3}{3} - 2t\right]_{\sqrt{2}}^{1}$$

$$= \frac{1-2\sqrt{2}}{3} - 2(1-\sqrt{2})$$

$$= \frac{4}{3}\sqrt{2} - \frac{5}{3} = \underline{\frac{4\sqrt{2}-5}{3}}.$$

$\boxed{3}$ $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \underset{\text{ビ}}{x}\cos\underset{\text{セ}}{\frac{x}{2}}dx$ 【2024 弘前大】

$$= \left[x\cdot 2\sin\frac{x}{2} - 1\cdot\left(-4\cos\frac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{2}{3}\pi}$$

$$= \frac{4\pi}{3}\underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}} + 4\underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}} - \frac{2\pi}{3}\underbrace{\sin\frac{\pi}{6}}_{\frac{1}{2}} - 4\underbrace{\cos\frac{\pi}{6}}_{\frac{\sqrt{3}}{2}}$$

$$= \frac{2\sqrt{3}}{3}\pi + 2 - \frac{\pi}{3} - 2\sqrt{3}$$

$$= \underline{\frac{2\sqrt{3}-1}{3}\pi - 2(\sqrt{3}-1).}$$

$\boxed{4}$ $\displaystyle\int_0^1 \frac{1}{x^2+x+1}dx$ 【2024 山梨大】

$$= \int_0^1 \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$

$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

と置換.

$dx = \frac{\sqrt{3}}{2}\cdot\frac{1}{\cos^2\theta}d\theta$

x	$0 \to 1$
θ	$\frac{\pi}{6} \to \frac{\pi}{3}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\frac{3}{4}(\tan^2\theta+1)}\cdot\frac{\sqrt{3}}{2\cos^2\theta}d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2}{\sqrt{3}}d\theta = \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6}$$

$$= \underline{\frac{\pi}{3\sqrt{3}}}$$

$\boxed{1}$ $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x(1+\cos x)}dx$ 【2025 弘前大】

$\cos x = t$.

$-\sin x\, dx = dt$

x	$0 \to \frac{\pi}{3}$
t	$1 \to \frac{1}{2}$

$$= \int_1^{\frac{1}{2}} \frac{-1}{t(1+t)}dt$$

$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$= \Big[\log t - \log(t+1)\Big]_{\frac{1}{2}}^{1}$$

$$= \log 1 - \log 2 - \log\frac{1}{2} + \log\frac{3}{2}$$

$$= \underline{\log\frac{3}{2}}.$$

$$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{4x^2-9x+6}{(x-1)(x-2)^2}$$

$$A(x-2)^2 + B(x-1)(x-2) + C(x-1) = 4x^2 - 9x + 6$$

$\boxed{2}$ $\displaystyle\int_3^4 \frac{4x^2-9x+6}{(x-1)(x-2)^2}dx$ 【2023 福島大】

$$= \int_3^4 \left(\frac{1}{x-1} + \frac{3}{x-2} + \frac{4}{(x-2)^2}\right)dx$$

$$= \Big[\log(x-1) + 3\log(x-2) - 4\cdot\frac{1}{x-2}\Big]_3^4$$

$$= \log 3 + 3\log 2 - 2 - \log 2 - 3\underbrace{\log 1}_{0} + 4$$

$$= \underline{\log 3 + 2\log 2 + 2}.$$

$\boxed{3}$ $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}d\theta$ 【2023 佐賀大】

$$= \Big[\log\left(\frac{1}{\cos\theta} + \tan\theta\right)\Big]_0^{\frac{\pi}{6}}$$

$$= \log\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = \log\sqrt{3} = \underline{\frac{1}{2}\log 3}.$$

$$\boxed{\begin{aligned}&\int \frac{1}{\cos x}dx = \log\left|\frac{1}{\cos x} + \tan x\right| + C \\ &\int \frac{1}{\sin x}dx = \log\left|\tan\frac{x}{2}\right| + C\end{aligned}}$$

は覚えておくと便利

$$\frac{1}{\sin x} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\frac{1}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}}{\tan\frac{x}{2}}$$ とみる！

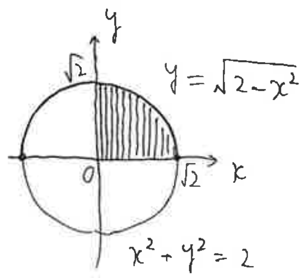
$\boxed{4}$ $\displaystyle\int_0^{\sqrt{2}} \sqrt{1-\frac{x^2}{2}}dx$ 【2023 愛知医科大】

$$= \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \sqrt{2-x^2}$$

$$= \frac{1}{\sqrt{2}} \times \pi(\sqrt{2})^2\cdot\frac{1}{4}$$

$$= \frac{\pi}{2\sqrt{2}}$$

$$= \underline{\frac{\sqrt{2}}{4}\pi}.$$



$\displaystyle\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$ は半径$\sqrt{2}$の四分円の面積.

$\boxed{1}$ $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\cos^2 x}dx$ 【2023 弘前大】

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} x\,(\tan x)'\,dx$$

部分積分

$$= \Big[\, x\tan x \,\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x\,dx$$

$$= \frac{\pi}{3}\cdot\sqrt{3} - \left(-\frac{\pi}{4}\right)\cdot(-1) + \Big[\, \log|\cos x| \,\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{3}\pi - \frac{\pi}{4} + \log\frac{1}{2} - \log\frac{\sqrt{2}}{2}$$

$$= \underline{\frac{\sqrt{3}}{3}\pi - \frac{\pi}{4} - \frac{1}{2}\log 2}.$$

$\boxed{2}$ $\displaystyle\int_{1}^{e} x^2\log x\,dx$ 【2023 産業医科大】

$$= \Big[\, \frac{x^3}{3}\log x \,\Big]_1^e - \int_1^e \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

$$= \frac{e^3}{3} - \int_1^e \frac{x^2}{3}\,dx$$

$$= \frac{e^3}{3} - \Big[\, \frac{x^3}{9} \,\Big]_1^e$$

$$= \underline{\frac{2}{9}e^3 + \frac{1}{9}}.$$

$\boxed{3}$ $\displaystyle\int_0^{\frac{\pi}{3}} \tan^3 x\,dx$ 【2022 学習院大】

$$= \int_0^{\frac{\pi}{3}} \tan x\left(\frac{1}{\cos^2 x} - 1\right)dx$$

$$= \int_0^{\frac{\pi}{3}} \left(\tan x\cdot\frac{1}{\cos^2 x} - \tan x\right)dx$$

$$= \Big[\, \frac{1}{2}(\tan x)^2 + \log|\cos x| \,\Big]_0^{\frac{\pi}{3}}$$

$$= \frac{3}{2} + \log\frac{1}{2} - \log 1$$

$$= \underline{\frac{3}{2} - \log 2}.$$

$\boxed{4}$ $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)\sqrt{1+x^2}}dx$ 【2025 群馬大】

$x = \tan\theta$
$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$
と置換

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{(1+\tan^2\theta)\cdot\frac{1}{\cos\theta}}\cdot\frac{d\theta}{\cos^2\theta}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos\theta} - \cos\theta\right)d\theta$$

$$= \Big[\, \log\left(\frac{1}{\cos\theta} + \tan\theta\right) - \sin\theta \,\Big]_0^{\frac{\pi}{4}}$$

$$= \underline{\log(\sqrt{2}+1) - \frac{\sqrt{2}}{2}}.$$

$\boxed{1}$ $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ 【2022 防衛大】

$$\left(\begin{array}{l} x = 2\sin\theta \quad \left(-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\right) \text{ とおく.} \\ dx = 2\cos\theta\, d\theta \qquad \begin{array}{c|c} x & 0 \to \sqrt{3} \\ \hline \theta & 0 \to \frac{\pi}{3} \end{array} \end{array}\right)$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4\cos^2\theta}} \cdot 2\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} 1\, d\theta = \underline{\frac{\pi}{3}}.$$

$\boxed{2}$ $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos x}{1+\cos x}}\, dx$ 【2025 京都大】

$$= \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}\, dx = \left[-2\log\left|\cos\frac{x}{2}\right|\right]_0^{\frac{\pi}{2}}$$

$$= 2\left(0 - \log\frac{1}{\sqrt{2}}\right) = \underline{\log 2}.$$

注

$$\cos 2\theta = \begin{cases} 1 - 2\sin^2\theta \\ 2\cos^2\theta - 1 \end{cases} \longrightarrow \begin{cases} 1 - \cos 2\theta = 2\sin^2\theta \\ 1 + \cos 2\theta = 2\cos^2\theta \end{cases}$$

より

$$\sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}} = \left|\tan\frac{x}{2}\right|.$$

$\displaystyle\sqrt{\frac{1-\cos x}{1+\cos x}} = \left|\frac{\sin x}{1+\cos x}\right|$ と変形してもよい.

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\, dx = \left[-\log(1+\cos x)\right]_0^{\frac{\pi}{2}}$$

$$= \underline{\log 2}.$$

$\boxed{3}$ $\displaystyle\int_0^{\frac{\pi}{4}} \underbrace{(x^2+1)}_{\text{び}}\underbrace{\cos 2x}_{\text{せ}}\, dx$ 【2021 学習院大】

$$= \left[(x^2+1)\cdot\frac{\sin 2x}{2} - 2x\cdot\frac{-\cos 2x}{4} + 2\cdot\frac{-\sin 2x}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\frac{x^2}{2} + \frac{1}{4}\right)\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\cdot\frac{\pi^2}{16} + \frac{1}{4} = \underline{\frac{\pi^2+8}{32}}$$

$\boxed{4}$ $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \log(\sin x)\, dx$ 【2021 福島大】

$$\left(\begin{array}{l} \sin x = t \text{ とおく.} \quad \cos x\, dx = dt \\ \qquad \begin{array}{c|c} x & \frac{\pi}{4} \to \frac{\pi}{2} \\ \hline t & \frac{1}{\sqrt{2}} \to 1 \end{array} \end{array}\right)$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \log t\, dt = \left[t\log t - t\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= -1 - \left(\frac{1}{\sqrt{2}}\log\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \underline{\frac{2 + \log 2 - 2\sqrt{2}}{2\sqrt{2}}}.$$

$$\left(= \frac{1}{2\sqrt{2}}\log 2 + \frac{1}{\sqrt{2}} - 1 \text{ などでも o.k.} \right)$$

$\boxed{1}$ $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\log(\sin x)}{\tan x}dx$ 【2020 横国大】

$\left(\begin{array}{l} \sin x = t \text{ とおく.} \quad \cos x\,dx = dt \\ x : \frac{\pi}{6} \to \frac{\pi}{4} \text{ のとき } t : \frac{1}{2} \to \frac{1}{\sqrt{2}} \end{array}\right)$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\log(\sin x)}{\sin x}\cos x\,dx = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{\log t}{t}dt$$

$$= \left[\frac{1}{2}(\log t)^2\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{2}\left\{\left(\log\frac{1}{\sqrt{2}}\right)^2 - \left(\log\frac{1}{2}\right)^2\right\}$$

$$= -\frac{3}{8}(\log 2)^2.$$

$\boxed{2}$ $\displaystyle\int_{\frac{1}{e}}^{1} (\log x + 1)^2 dx$ 【2020 岐阜大】

$\left(\begin{array}{l} \log x = t \text{ とおく.} \quad \frac{1}{x}dx = dt \\ \begin{array}{c|c} x & \frac{1}{e} \to 1 \\ \hline t & -1 \to 0 \end{array} \qquad dx = e^t dt \end{array}\right)$

$$= \int_{-1}^{0} \underset{び}{(t+1)^2}\underset{せ}{e^t}\,dt$$

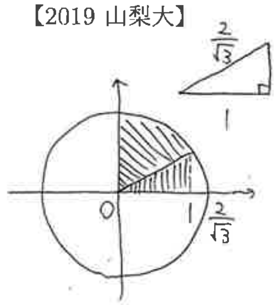
$$= \left[(t+1)^2 e^t - 2(t+1)e^t + 2e^t\right]_{-1}^{0}$$

$$= \left[(t^2+1)e^t\right]_{-1}^{0}$$

$$= 1 - \frac{2}{e}.$$

$\boxed{3}$ $\displaystyle\int_{0}^{1} \sqrt{4-3x^2}\,dx$ 【2019 山梨大】

$$= \sqrt{3}\underbrace{\int_0^1 \sqrt{\frac{4}{3} - x^2}\,dx}_{S}$$



$$S = \pi\left(\frac{2}{\sqrt{3}}\right)^2 \times \frac{1}{6} + \frac{1}{2}\cdot 1 \cdot \frac{1}{\sqrt{3}}$$

$$= \frac{2}{9}\pi + \frac{1}{2\sqrt{3}}$$

より, $\displaystyle\int_0^1 \sqrt{4-3x^2}\,dx = \frac{2\sqrt{3}}{9}\pi + \frac{1}{2}.$

(注) $x = \frac{2}{\sqrt{3}}\sin\theta \left(-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\right)$ とおいてもよい.

$$\int_0^1 \sqrt{4-3x^2}\,dx = \frac{4}{\sqrt{3}}\int_0^{\frac{\pi}{3}} \cos^2\theta\,d\theta = \frac{2}{\sqrt{3}}\int_0^{\frac{\pi}{3}} (1+\cos 2\theta)\,d\theta$$

$$= \frac{2}{\sqrt{3}}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} = \frac{2\sqrt{3}}{9}\pi + \frac{1}{2}.$$

$\boxed{4}$ $\displaystyle\int_{0}^{2\sqrt{3}} \frac{dx}{4+x^2}$ 【2020 福岡大】

$\left(\begin{array}{l} x = 2\tan\theta \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \text{ とおく.} \\ \begin{array}{c|c} x & 0 \to 2\sqrt{3} \\ \hline \theta & 0 \to \frac{\pi}{3} \end{array} \qquad dx = \frac{2}{\cos^2\theta}d\theta \end{array}\right)$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{4(1+\tan^2\theta)}\cdot\frac{2}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{2}\,d\theta = \frac{1}{2}\cdot\frac{\pi}{3} = \frac{\pi}{6}.$$

$\boxed{1}$ $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^4 x\,dx$ 【2019 産業医科大】

$$= 2\int_0^{\frac{\pi}{4}} \tan^2 x \cdot \tan^2 x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \tan^2 x\left(\frac{1}{\cos^2 x} - 1\right)dx$$

$$= 2\int_0^{\frac{\pi}{4}} \left(\tan^2 x \cdot \frac{1}{\cos^2 x} + 1 - \frac{1}{\cos^2 x}\right)dx$$

$$= 2\left[\frac{1}{3}\tan^3 x + x - \tan x\right]_0^{\frac{\pi}{4}}$$

$$= 2\left(\frac{1}{3} + \frac{\pi}{4} - 1\right) = \underline{\frac{\pi}{2} - \frac{4}{3}}.$$

$\boxed{2}$ $\displaystyle\int_{-2}^{-1} \sqrt{\frac{x+1}{x-1}}\,dx$ 【2018 宮城教育大】

$\sqrt{\dfrac{x+1}{x-1}} = t$ とおく． $x = 1 + \dfrac{2}{t^2-1}$．

$$\begin{array}{c|c} x & -2 \to -1 \\ \hline t & \frac{1}{\sqrt{3}} \to 0 \end{array} \qquad dx = \frac{-4t}{(t^2-1)^2}\,dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} t \cdot \frac{4t}{(t^2-1)^2}\,dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \left\{\frac{1}{(t-1)^2} + \frac{1}{t-1} - \frac{1}{t+1} + \frac{1}{(t+1)^2}\right\}dt$$

$$= -\left[\frac{1}{t-1} + \log\left|\frac{t+1}{t-1}\right| + \frac{1}{t+1}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \underline{\sqrt{3} - \log(2+\sqrt{3})}.$$

$\boxed{3}$ $\displaystyle\int_1^e \sqrt{x}\log x\,dx$ 【2018 大阪府立大】

そのまま部分積分してもよいが，$\log x = t$ とおいて指数関数にすれば，反復部分積分にもちこめる！

$$x = e^t,\quad dx = e^t\,dt,\quad \begin{array}{c|c} x & 1 \to e \\ \hline t & 0 \to 1 \end{array}$$

$$= \int_0^1 e^{\frac{t}{2}} \cdot t \cdot e^t\,dt = \int_0^1 t\cdot e^{\frac{3}{2}t}\,dt$$

$$= \left[t\cdot\frac{2}{3}e^{\frac{3}{2}t} - 1\cdot\frac{4}{9}e^{\frac{3}{2}t}\right]_0^1.$$

$$= \left[\frac{6t-4}{9}e^{\frac{3}{2}t}\right]_0^1 = \underline{\frac{2}{9}e^{\frac{3}{2}} + \frac{4}{9}}.$$

$\boxed{4}$ $\displaystyle\int_0^{\frac{\pi}{3}} \sqrt{1+\sin x}\,dx$ 【2018 宮崎大】

$$1 + \sin x = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2$$

より，

$$\sqrt{1+\sin x} = \left|\sin\frac{x}{2} + \cos\frac{x}{2}\right|.$$

$$= \int_0^{\frac{\pi}{3}} \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)dx$$

$$= \left[2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \underline{3 - \sqrt{3}}.$$

注 他にも様々な計算が可能．

$= \displaystyle\int_0^{\frac{\pi}{3}} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$ とみて $\sin x = t$ と置換

$\boxed{1}$ $\displaystyle\int_0^{-1} \frac{x^2+2x+1}{\sqrt{-x^2-2x+1}}dx$ 【2018 産業医科大】

$x+1=\sqrt{2}\sin\theta$

$\left(-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\right)$

x	$0 \to -1$
θ	$\frac{\pi}{4} \to 0$

$$= \int_0^{-1} \frac{(x+1)^2}{\sqrt{2-(x+1)^2}}dx$$

$$= \int_{\frac{\pi}{4}}^{0} \frac{2\sin^2\theta}{\sqrt{2}\cos\theta}\cdot\sqrt{2}\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{0} (1-\cos 2\theta)\, d\theta$$

$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{0}$$

$$= \underline{\frac{1}{2} - \frac{\pi}{4}}.$$

$\boxed{2}$ $\displaystyle\int_0^1 \frac{4x+1}{x^4+2x^3+x+2}dx$ 【2017 東京理科大】

$x-\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$

$$= \int_0^1 \left(\frac{1}{x+2} - \frac{1}{x+1} + \frac{1}{x^2-x+1}\right)dx$$

$$= \left[\log\left|\frac{x+2}{x+1}\right|\right]_0^1 + \int_0^1 \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$= \log\frac{3}{2} - \log 2 + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{}{\frac{3}{4}(1+\tan^2\theta)}\cdot\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{\cos^2\theta}$$

$$= \log\frac{3}{4} + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2}{\sqrt{3}}\, d\theta$$

$$= \log\frac{3}{4} + \frac{2}{\sqrt{3}}\cdot\frac{\pi}{3}$$

$$= \underline{\frac{2\sqrt{3}}{9}\pi - \log\frac{4}{3}}.$$

$\boxed{3}$ $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{\cos 2x}dx$ 【2017 横浜市立大】

$\cos x = t$ とおく

$-\sin x\, dx = dt$

x	$\frac{\pi}{3} \to \frac{\pi}{2}$
t	$\frac{1}{2} \to 0$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{2\cos^2 x - 1}dx$$

$$= \int_{\frac{1}{2}}^{0} \frac{-1}{2t^2-1}dt = \int_0^{\frac{1}{2}} \frac{1}{2t^2-1}dt$$

$$= \int_0^{\frac{1}{2}} \frac{1}{2}\cdot\frac{1}{t^2-\frac{1}{2}}dt = \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\left(t+\frac{1}{\sqrt{2}}\right)\left(t-\frac{1}{\sqrt{2}}\right)}dt$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} \left(\frac{-\frac{1}{\sqrt{2}}}{t+\frac{1}{\sqrt{2}}} + \frac{\frac{1}{\sqrt{2}}}{t-\frac{1}{\sqrt{2}}}\right)dt = \frac{1}{2\sqrt{2}}\left[\log\left|\frac{t-\frac{1}{\sqrt{2}}}{t+\frac{1}{\sqrt{2}}}\right|\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right|\right]_0^{\frac{1}{2}} = \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}-2}{\sqrt{2}+2}\right|$$

$$= \frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \underline{\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)}.$$

$\boxed{4}$ $\displaystyle\int_0^{\frac{\pi}{2}} \left(\cos\theta\sin^3\theta - \cos^3\theta\sin^3\theta + \cos\theta\sin^7\theta\right)d\theta$ 【2018 防衛医科大】

$$\underbrace{\underbrace{\cos\theta\sin^3\theta - \cos^3\theta\sin^3\theta}_{\begin{gathered}= \cos\theta\sin^3\theta(1-\cos^2\theta) \\ = \cos\theta\sin^5\theta\end{gathered}} + \cos\theta\sin^7\theta}_{= \cos\theta(\sin^5\theta+\sin^7\theta)}$$

$\sin\theta = t$

$$= \int_0^1 (t^5+t^7)\,dt$$

$$= \left[\frac{1}{6}t^6 + \frac{1}{8}t^8\right]_0^1$$

$$= \frac{1}{6}+\frac{1}{8} = \underline{\frac{7}{24}}.$$

$\boxed{1}$ $\displaystyle\int_e^{e^2} \frac{\log(\log x)}{x\log x}dx$ 【2019 秋田大】

$\log x = t$ とおく

$\frac{1}{x}dx = dt$

x	$e \to e^2$
t	$1 \to 2$

$$= \int_1^2 \frac{\log t}{t}\,dt$$

$$= \left[\frac{1}{2}(\log t)^2\right]_1^2$$

$$= \frac{(\log 2)^2}{2}.$$

$\boxed{2}$ $\displaystyle\int_1^{1+\sqrt{3}} \frac{x^3}{x^2-2x+2}dx$ 【2017 福島県立医科大】

$$= \int_1^{1+\sqrt{3}} \left\{ x+2+\frac{2x-2}{x^2-2x+2} - \frac{2}{(x-1)^2+1} \right\} dx$$

$$= \left[\frac{1}{2}(x+2)^2 + \log(x^2-2x-2)\right]_1^{1+\sqrt{3}} - 2\int_0^{\frac{\pi}{3}} 1\cdot d\theta$$

$x-1 = \tan\theta$

$$= \frac{1}{2}(3+\sqrt{3})^2 + \log 4 - \frac{1}{2}\cdot 3^2 - 2\cdot\frac{\pi}{3}$$

$$= \frac{(6+\sqrt{3})\cdot\sqrt{3}}{2} + 2\log 2 - \frac{2}{3}\pi$$

$$= \frac{3}{2} + 3\sqrt{3} + 2\log 2 - \frac{2\pi}{3}.$$

$\boxed{3}$ $\displaystyle\int_0^{\frac{\pi}{3}} x\sin^2(2x)dx$ 【2016 宮崎大】

$$= \int_0^{\frac{\pi}{3}} x\cdot\frac{1-\cos 4x}{2}dx = \int_0^{\frac{\pi}{3}}\frac{x}{2}dx - \frac{1}{2}\int_0^{\frac{\pi}{3}} x\cos 4x\,dx.$$

ここで,

$$\int_0^{\frac{\pi}{3}} x\cos 4x\,dx = \left[x\cdot\frac{\sin 4x}{4} - 1\cdot\frac{-\cos 4x}{16}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3}\cdot\frac{-\frac{\sqrt{3}}{2}}{4} + \frac{1}{16}\left(-\frac{1}{2}\right) - \frac{1}{16}$$

$$= -\frac{\sqrt{3}}{24}\pi - \frac{3}{32}$$

より,

$$\int_0^{\frac{\pi}{3}} x\sin^2(2x)\,dx = \left[\frac{x^2}{4}\right]_0^{\frac{\pi}{3}} - \frac{1}{2}\left(-\frac{\sqrt{3}}{24}\pi - \frac{3}{32}\right)$$

$$= \frac{\pi^2}{36} + \frac{\sqrt{3}}{48}\pi + \frac{3}{64}.$$

$\boxed{4}$ $\displaystyle\int_1^2 (\log x)^3 dx$ 【2016 奈良県立医科大】

$\log x = t$ とおく $e^t = x$

x	$1 \to 2$
t	$0 \to \log 2$

$dx = e^t dt$

$$= \int_0^{\log 2} t^3 e^t\,dt$$

$$= \left[(t^3-3t^2+6t-6)e^t\right]_0^{\log 2}$$

$$= \left\{(\log 2)^3 - 3(\log 2)^2 + 6\log 2 - 6\right\}\times 2 + 6$$

$$= 2(\log 2)^3 - 6(\log 2)^2 + 12\log 2 - 6.$$

$\boxed{1}$ $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1-2x^2}{x(1-x^2)}dx$ 【2001 山形大】

$\left(x = \sin\theta \quad \left(-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\right) \text{ とおく.} \right)$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1-2\sin^2\theta}{\sin\theta \cdot \cos^2\theta} \cdot \cos\theta \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\cos 2\theta}{\sin 2\theta} d\theta$$

$$= \Big[\log(\sin 2\theta) \Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \underline{\log\frac{\sqrt{3}}{2}}$$

$\boxed{3}$ $\displaystyle\int_0^{\frac{\pi}{2}} (\cos x)\log(3-2\sin x)dx$ 【1997 奈良県立医科大】

$\left(\begin{array}{l} 3 - 2\sin x = t \text{ とおく.} \\ -2\cos x \, dx = dt \end{array} \quad \begin{array}{c|c} x & 0 \to \frac{\pi}{2} \\ \hline t & 3 \to 1 \end{array} \right)$

$$= \int_3^1 -\frac{1}{2}\log t \, dt$$

$$= \frac{1}{2}\Big[t\log t - t \Big]_1^3$$

$$= \frac{1}{2}\left(3\log 3 - 3 - 0 + 1 \right)$$

$$= \underline{\frac{3}{2}\log 3 - 1}.$$

$\boxed{2}$ $\displaystyle\int_1^8 e^{-\sqrt{x}}dx$ 【1999 浜松医科大】

$\left(\sqrt{x} = t \text{ とおく.} \quad x = t^2 \quad dx = 2t\,dt \right)$

$$= \int_1^{2\sqrt{2}} e^{-t} \cdot 2t \, dt$$

$$= \left[\frac{e^{-t}}{-1} \cdot 2t - e^{-t} \cdot 2 \right]_1^{2\sqrt{2}}$$

$$= \Big[(2t+2)e^{-t} \Big]_{2\sqrt{2}}^{1}$$

$$= 4e^{-1} - (4\sqrt{2}+2)e^{-2\sqrt{2}}$$

$$= \underline{\frac{4}{e} - \frac{4\sqrt{2}+2}{e^{2\sqrt{2}}}}.$$

$\boxed{4}$ $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{dx}{\sin x}$ 【2000 東京農工大】

$$= \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{\sin x}{1-\cos^2 x} dx = \int_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}} \frac{-1}{1-t^2} dt$$

$$= 2\int_0^{\frac{1}{\sqrt{2}}} \frac{-1}{(t+1)(t-1)} dt$$

$$= 2\int_0^{\frac{1}{\sqrt{2}}} \left(\frac{\frac{1}{2}}{t+1} + \frac{-\frac{1}{2}}{t-1} \right) dt$$

$$= \left[\log\left| \frac{t+1}{t-1} \right| \right]_0^{\frac{1}{\sqrt{2}}} = \log\frac{\sqrt{2}+1}{\sqrt{2}-1} = \underline{2\log(\sqrt{2}+1)}.$$

(別) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{dx}{\sin x} = \left[\log\left|\tan\frac{x}{2}\right| \right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi}$

$$= \log\left(\tan\frac{3}{8}\pi\right) - \log\left(\tan\frac{\pi}{8}\right) = \log\frac{\tan\frac{3}{8}\pi}{\tan\frac{\pi}{8}}$$

$$= \log(3+2\sqrt{2}) = 2\log(1+\sqrt{2}).$$

1 $\displaystyle\int_1^3 (x-1)(x-3)^7 dx$ 【2001 関西大】

$x-3=t$ とおく

$$= \int_{-2}^{0} (t+2)\, t^7\, dt$$

$$= \int_{-2}^{0} (t^8 + 2t^7)\, dt$$

$$= \left[\frac{1}{9}t^9 + \frac{1}{4}t^8 \right]_{-2}^{0}$$

$$= 0 - (-2)^8 \cdot \left(\frac{-2}{9} + \frac{1}{4} \right) = -\frac{64}{9}.$$

別 部分積分法

$$= \left[(x-1)\cdot\frac{(x-3)^8}{8} - 1\cdot\frac{(x-3)^9}{72} \right]_1^3$$

$$= 0 + \frac{(-2)^9}{72} = -\frac{64}{9}.$$

2 $\displaystyle\int_0^{\frac{\pi}{2}} e^{\sin x}\sin 2x\, dx$ 【2000 信州大】

$$= \int_0^{\frac{\pi}{2}} e^{\sin x}\cdot 2\sin x\cos x\, dx$$

$$= \int_0^1 e^t \cdot 2t\, dt$$

$$= \left[e^t \cdot 2t - e^t \cdot 2 \right]_0^1$$

$$= \left[2(t-1)e^t \right]_0^1$$

$$= 0 - 2\cdot(-1)e^0$$

$$= 2.$$

3 $\displaystyle\int_0^{\frac{\pi}{3}} \underbrace{\sin x\sin 2x\cos^2 x}_{=\,2\sin^2 x\cos^3 x}\, dx$ 【2011 津田塾大】

$$= \int_0^{\frac{\pi}{3}} 2\sin^2 x\,(1-\sin^2 x)\cos x\, dx$$

$\sin x = t$ とおく

$$= \int_0^{\frac{\sqrt{3}}{2}} 2t^2(1-t^2)\, dt$$

$$= 2\left[\frac{1}{3}t^3 - \frac{1}{5}t^5 \right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{2}{15}\left[t^3(5-3t^2) \right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{11}{80}\sqrt{3}.$$

4 $\displaystyle\int_0^1 \frac{x^2+5}{(x+1)^2(x-2)}dx$ 【2011 同志社大】

$$= \int_0^1 \left(\frac{1}{x-2} - \frac{2}{(x+1)^2} \right) dx$$

$$= \left[\log|x-2| + \frac{2}{x+1} \right]_0^1$$

$$= (\log 1 + 1) - (\log 2 + 2)$$

$$= -1 - \log 2.$$

メモ

$$\frac{x^2+5}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

$$x^2+5 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$$

$x=-1,\ 2,\ 0$ を代入し，A, B, C を決定

$\boxed{1}$ $\displaystyle\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$ 【2008 大阪市立大】

$\sin^2 x + \cos^2 x = 1$

$\div\cos^2 x$

$\tan^2 x + 1 = \dfrac{1}{\cos^2 x}$

$$= \int_0^{\frac{\pi}{4}} x\left(\frac{1}{\cos^2 x} - 1\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\underset{ビ}{x}\cdot \underset{セ}{\frac{1}{\cos^2 x}} - x\right)dx$$

$$= \left[x\cdot\tan x - 1\cdot(-\log|\cos x|) - \frac{1}{2}x^2\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{\pi^2}{16}$$

$$= \underline{\frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\log 2}\ .$$

$\boxed{2}$ $\displaystyle\int_1^{e} x(\log x)^2\,dx$ 【2006 横浜国立大】

$\left(\ \log x = t\right.$ とおく． $x = e^t$ $\quad dx = e^t\,dt$

x	$1 \to e$
t	$0 \to 1$

$\left.\right)$

$$= \int_0^1 e^t\cdot t^2 e^t\,dt$$

$$= \int_0^1 \underset{セ}{e^{2t}}\cdot \underset{ビ}{t^2}\,dt$$

$$= \left[\frac{e^{2t}}{2}\cdot t^2 - \frac{e^{2t}}{4}\cdot 2t + \frac{e^{2t}}{8}\cdot 2\right]_0^1$$

$$= \left[\frac{2t^2-2t+1}{4}\,e^{2t}\right]_0^1$$

$$= \underline{\frac{e^2-1}{4}}\ .$$

$\boxed{3}$ $\displaystyle\int_{\frac{\pi}{12}}^{\frac{5}{12}\pi} \frac{\sin^2 x}{\sin x + \cos x}dx$ 【1992 北見工業大】

$x + \frac{\pi}{4} = t$ とおく．

x	$\frac{\pi}{12} \to \frac{5}{12}\pi$
t	$\frac{\pi}{3} \to \frac{2}{3}\pi$

$$= \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \frac{\sin^2\left(t-\frac{\pi}{4}\right)}{\sqrt{2}\sin t}dt$$

$$= \frac{1}{2\sqrt{2}}\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \frac{1-\cos\left(2t-\frac{\pi}{2}\right)}{\sin t}dt$$

$$= \frac{1}{2\sqrt{2}}\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \left(\frac{1}{\sin t} - 2\cos t\right)dt$$

$$= \frac{1}{2\sqrt{2}}\left[\ \log\left|\tan\frac{x}{2}\right| - 2\sin t\right]_{\frac{\pi}{3}}^{\frac{2}{3}\pi}$$

$$= \frac{1}{2\sqrt{2}}\left(\log\sqrt{3} - \log\frac{1}{\sqrt{3}}\right) = \underline{\frac{1}{2\sqrt{2}}\log 3}\ .$$

注 $\displaystyle\int\frac{1}{\sin\theta}d\theta = \log\left|\tan\frac{\theta}{2}\right| + C$ は覚えておこう！

$\boxed{4}$ $\displaystyle\int_0^{\frac{\pi}{2}} x\sin^3 x\,dx$ 【2000 東京農工大】

$\sin 3x = 3\sin x - 4\sin^3 x$

$\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \underset{ビ}{x}\,\underset{セ}{(3\sin x - \sin 3x)}\,dx$$

$$= \frac{1}{4}\left[x\cdot\left(-3\cos x + \frac{\cos 3x}{3}\right) - 1\cdot\left(-3\sin x + \frac{\sin 3x}{9}\right)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left\{\frac{\pi}{4}\left(-3\cdot\frac{\sqrt{2}}{2} + \frac{1}{3}\left(-\frac{\sqrt{2}}{2}\right)\right) + 3\cdot\frac{\sqrt{2}}{2} - \frac{1}{9}\cdot\frac{\sqrt{2}}{2}\right\}$$

$$= \underline{\left(\frac{13}{36} - \frac{5\pi}{48}\right)\sqrt{2}}\ .$$

$\boxed{1}$ $\displaystyle\int_0^{\log\sqrt{3}} \frac{1}{e^x+e^{-x}}dx$. 【1994 愛媛大】

$t = e^x$ とおく

$dt = e^x\,dx$

$dx = \frac{1}{t}dt$

x	$0 \to \log\sqrt{3}$
t	$1 \to \sqrt{3}$

$$= \int_1^{\sqrt{3}} \frac{1}{t+\frac{1}{t}}\cdot\frac{1}{t}\,dt$$

$$= \int_1^{\sqrt{3}} \frac{1}{t^2+1}\,dt$$

$t = \tan\theta$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta+1}\cdot\frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1\,d\theta = \frac{\pi}{3}-\frac{\pi}{4} = \frac{\pi}{12}.$$

$\boxed{2}$ $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^2x+3\cos^2x}$ 【2006 横浜国立大】

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2x+3}\cdot\frac{1}{\cos^2x}\,dx$$

$\tan x = t$

$$= \int_0^1 \frac{1}{t^2+3}\,dt$$

$t = \sqrt{3}\tan\theta$ $\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{3(1+\tan^2\theta)}\cdot\frac{\sqrt{3}}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}}{3}\,d\theta$$

$$= \frac{\sqrt{3}}{3}\times\frac{\pi}{6} = \frac{\sqrt{3}}{18}\pi.$$

$\boxed{3}$ $\displaystyle\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^4 x}dx$ 【2001 東京理科大】

$\tan x = t$ とおく. $\frac{1}{\cos^2x}dx = dt$

x	$0 \to \frac{\pi}{4}$
t	$0 \to 1$

$1+\tan^2x = \frac{1}{\cos^2x}$

$$= \int_0^1 e^t(1+t^2)\,dt$$

$$= \Big[e^t\{(1+t^2)-2t+2\}\Big]_0^1$$

$$= \Big[e^t(t^2-2t+3)\Big]_0^1$$

$$= 2e-3$$

$\cos 3x = 4\cos^3x - 3\cos x$

$\longrightarrow\ \cos^3x = \frac{3\cos x+\cos 3x}{4}$

$\boxed{4}$ $\displaystyle\int_0^{\frac{\pi}{2}} x\cos^3x\,dx$ 【2000 法政大】

$$= \int_0^{\frac{\pi}{2}} x\cdot\frac{3\cos x+\cos 3x}{4}\,dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} x(3\cos x+\cos 3x)\,dx$$

$$= \frac{1}{4}\left[x\cdot\left(3\sin x+\frac{\sin 3x}{3}\right)-1\cdot\left(-3\cos x-\frac{\cos 3x}{9}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left\{\frac{\pi}{2}\left(3+\left(-\frac{1}{3}\right)\right)+\left(-3-\frac{1}{9}\right)\right\}$$

$$= \frac{\pi}{3}-\frac{7}{9} = \frac{3\pi-7}{9}.$$

$\boxed{1}$ $\displaystyle\int_0^{2\pi}\sqrt{1-\cos x}\,dx$ 【1998 東京電気大】

$$= \int_0^{2\pi}\sqrt{\frac{1}{2}\sin^2\frac{x}{2}}\,dx = \frac{1}{\sqrt{2}}\int_0^{2\pi}\sin\frac{x}{2}\,dx$$

$$= -\sqrt{2}\left[\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= \underline{2\sqrt{2}}.$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta \longrightarrow 2\sin^2\theta = 1-\cos 2\theta$$

$$= 2\cos^2\theta - 1$$

$\boxed{2}$ $\displaystyle\int_1^4 \frac{dx}{\sqrt{3-\sqrt{x}}}$ 【2001 横浜国立大】

($3-\sqrt{x} = t$ とおく.
$x = (t-3)^2$

x	$1 \to 4$
t	$2 \to 1$

)

$$= \int_2^1 \frac{1}{\sqrt{t}}\cdot 2(t-3)\,dt$$

$$= 2\int_1^2\left(\frac{3}{\sqrt{t}} - \sqrt{t}\right)dt$$

$$= 2\left[3\cdot 2t^{\frac{1}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right]_1^2$$

$$= 12(\sqrt{2}-1) - \frac{4}{3}(2\sqrt{2}-1)$$

$$= \underline{\frac{28}{3}\sqrt{2} - \frac{32}{3}}.$$

$\boxed{3}$ $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{1+x-2x^2}dx$ 【2007 防衛大】

$$= -\frac{1}{2}\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{(x+\frac{1}{2})(x-1)}\,dx$$

$$= -\frac{1}{2}\int_{-\frac{1}{3}}^{\frac{1}{3}}\left(\frac{-\frac{2}{3}}{x+\frac{1}{2}} + \frac{\frac{2}{3}}{x-1}\right)dx$$

$$= -\frac{1}{3}\left[\log\left|\frac{x-1}{x+\frac{1}{2}}\right|\right]_{-\frac{1}{3}}^{\frac{1}{3}}$$

$$= \frac{1}{3}\left(\log 8 - \log\frac{4}{5}\right) = \underline{\frac{\log 10}{3}}.$$

$\boxed{4}$ $\displaystyle\int_0^{\pi} x\cos\left(x+\frac{\pi}{3}\right)dx$ 【2009 茨城大】

$$= \left[x\cdot\sin\left(x+\frac{\pi}{3}\right) - 1\cdot\left\{-\cos\left(x+\frac{\pi}{3}\right)\right\}\right]_0^{\pi}$$

$$= \pi\sin\frac{4}{3}\pi + \cos\frac{4}{3}\pi - 0 - \cos\frac{\pi}{3}$$

$$= \pi\cdot\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{2} - \frac{1}{2}$$

$$= \underline{-\frac{\sqrt{3}}{2}\pi - 1}.$$

$\boxed{1}$ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{1+\cos^2 x}}\,dx$ 【1998 東京電気大】

$$= \int_0^{\frac{\pi}{2}} \frac{2\sin x\cos x}{\sqrt{1+\cos^2 x}}\,dx$$

$\cos x = t$

$-\sin x\,dx = dt$

$$= \int_1^0 \frac{-2t}{\sqrt{1+t^2}}\,dt$$

$$= \int_0^1 \frac{2t}{\sqrt{1+t^2}}\,dt$$

$$= \left[\,2(1+t^2)^{\frac{1}{2}}\,\right]_0^1$$

$$= \underline{2\sqrt{2}-2}\,.$$

$\boxed{2}$ $\displaystyle\int_1^e \frac{(\log x)^2}{\sqrt{x}}\,dx$ 【2001 横浜国立大】

$\log x = t$

$x = e^t$

$$= \int_0^1 \frac{t^2}{e^{\frac{t}{2}}}\,e^t\,dt$$

$$= \int_0^1 t^2 e^{\frac{t}{2}}\,dt$$

$$= \left[\,t^2\cdot 2e^{\frac{t}{2}} - 2t\cdot 4e^{\frac{t}{2}} + 2\cdot 8e^{\frac{t}{2}}\,\right]_0^1$$

$$= \left[\,2(t^2-4t+8)e^{\frac{t}{2}}\,\right]_0^1$$

$$= 10e^{\frac{1}{2}} - 16$$

$$= \underline{10\sqrt{e}-16}\,.$$

$\boxed{3}$ $\displaystyle\int_0^{\frac{\pi}{6}} \left(\sin\frac{3x}{2}\right)\left(\cos\frac{x}{2}\right)dx$ 【2008 関西大】

↓ 積・和公式

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(\sin 2x + \sin x)\,dx$$

$$= \frac{1}{2}\left[\,-\frac{1}{2}\cos 2x - \cos x\,\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2} + 1\right)$$

$$= \underline{\frac{5}{8} - \frac{\sqrt{3}}{4}}$$

$\boxed{4}$ $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2}\log\sqrt{1+x^2}\,dx$ 【2012 京都大】

$$= \frac{1}{2}\int_1^{\sqrt{3}} \frac{1}{x^2}\log(1+x^2)\,dx$$

$$= \frac{1}{2}\left\{\left[\,-\frac{1}{x}\log(1+x^2)\,\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} -\frac{1}{x}\cdot\frac{2x}{1+x^2}\,dx\right\}$$

$$= \frac{1}{2}\left\{-\frac{1}{\sqrt{3}}\log 4 + \log 2 + \int_1^{\sqrt{3}} \frac{2}{1+x^2}\,dx\right\}$$

$$= \frac{1}{2}\left(1-\frac{2}{\sqrt{3}}\right)\log 2 + \int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx$$

$x = \tan\theta$

$$= \left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\log 2 + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1\,d\theta$$

$$= \underline{\frac{\pi}{12} - \left(\frac{1}{\sqrt{3}}-\frac{1}{2}\right)\log 2}\,.$$

$\boxed{1}$ $\displaystyle\int_0^{\frac{\pi}{2}} \sin(5x)\cos(2x)dx$ 【2019 宮崎大】

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 7x + \sin 3x}{2}\, dx$$

$$= -\frac{1}{2}\left[\frac{\cos 7x}{7} + \frac{\cos 3x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{1}{7} + \frac{1}{3}\right)$$

$$= \frac{5}{21}.$$

$\boxed{2}$ $\displaystyle\int_0^1 \sqrt{1+2\sqrt{x}}\,dx$ 【2013 横浜国立大】

$\left(1 + 2\sqrt{x} = t \text{ とおく. } \sqrt{x} = \frac{t-1}{2}. \quad x = \frac{(t-1)^2}{4}. \quad dx = \frac{t-1}{2}dt. \right)$

$$= \int_1^3 \sqrt{t}\cdot\frac{t-1}{2}\,dt = \frac{1}{2}\int_1^3 \left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right)dt$$

$$= \frac{1}{2}\left[\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right]_1^3$$

$$= \frac{1}{5}\left(3^{\frac{5}{2}} - 1\right) - \frac{1}{3}\left(3^{\frac{3}{2}} - 1\right)$$

$$= \frac{1}{5}\left(9\sqrt{3} - 1\right) - \frac{1}{3}\left(3\sqrt{3} - 1\right)$$

$$= \frac{4}{5}\sqrt{3} + \frac{2}{15}.$$

$\boxed{3}$ $\displaystyle\int_0^1 \left(x^2 + \frac{x}{\sqrt{1+x^2}}\right)\left(1 + \frac{x}{(1+x^2)\sqrt{1+x^2}}\right)dx$ 【2019 東京大】

$\left(x = \tan\theta \ \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \text{ とおく. } \quad \frac{1}{\sqrt{1+x^2}} = \cos\theta, \quad dx = \frac{1}{\cos^2\theta}d\theta \right)$

$$= \int_0^{\frac{\pi}{4}} \left(\tan^2\theta + \sin\theta\right)\left(1 + \sin\theta\cos^2\theta\right)\frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left\{\left(\tan^2\theta\cdot\frac{1}{\cos^2\theta} + \underbrace{\sin^2\theta}_{=\frac{1-\cos 2\theta}{2}}\right) + \sin\theta\underbrace{\left(\frac{1}{\cos^2\theta} + \tan^2\theta\right)}_{=\frac{1+\sin^2\theta}{\cos^2\theta} = \frac{2-\cos^2\theta}{\cos^2\theta}}\right\}d\theta \qquad \cos\theta = t$$

$$= \left[\frac{1}{3}\tan^3\theta + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{\frac{\pi}{4}} + \int_1^{\frac{1}{\sqrt{2}}}\left(\frac{2}{t^2} - 1\right)(-1)\,dt$$

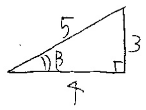
$$= \frac{1}{3} + \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) + \left[t + \frac{2}{t}\right]_1^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{3} + \frac{\pi}{8} - \frac{1}{4} + \frac{1}{\sqrt{2}} + 2\sqrt{2} - 1 - 2$$

$$= \frac{\pi}{8} - \frac{35}{12} + \frac{5}{2}\sqrt{2}.$$

$\boxed{1}$ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3\sin x + 4\cos x}$ 【2017 横浜国立大】

$$3\sin x + 4\cos x = 5\left(\cos x \cdot \frac{4}{5} + \sin x \cdot \frac{3}{5}\right)$$



$$= 5\cos(x-\beta)$$

$$\cos\beta = \frac{4}{5}, \quad \sin\beta = \frac{3}{5}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{5\cos(x-\beta)}\,dx = \int_{-\beta}^{\frac{\pi}{2}-\beta} \frac{1}{5\cos t}\,dt$$

$$= \frac{1}{5}\left[\log\left|\frac{1+\sin t}{\cos t}\right|\right]_{-\beta}^{\frac{\pi}{2}-\beta}$$

$$= \frac{1}{5}\left(\log\frac{1+\cos\beta}{\sin\beta} - \log\frac{1-\sin\beta}{\cos\beta}\right)$$

$$= \frac{1}{5}\left(\log\frac{9}{3} - \log\frac{2}{4}\right) = \underline{\frac{\log 6}{5}}$$

$$\int \frac{1}{\cos\theta}\,d\theta = \log\left|\frac{1}{\cos\theta} + \tan\theta\right| + C$$

は覚えておくと便利!!

$\boxed{2}$ $\displaystyle\int_0^{\sqrt{3}} \frac{x\sqrt{x^2+1} + 2x^3 + 1}{x^2+1}\,dx$ 【2025 京都大】

$x = \tan\theta$ と置換

$$= \int_0^{\sqrt{3}} \left(\frac{x}{\sqrt{x^2+1}} + 2x - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)dx$$

$$= \left[\sqrt{x^2+1} + x^2 - \log(x^2+1)\right]_0^{\sqrt{3}} + \int_0^{\frac{\pi}{3}} d\theta$$

$$= \underline{4 - 2\log 2 + \frac{\pi}{3}}$$

$\boxed{3}$ $\displaystyle\int_1^{512} \frac{\sin(\pi\log_2 x)}{x}\,dx$ 【1993 横浜国立大】

$\pi\log_2 x = t$ とおく.

$$t = \pi \cdot \frac{\log x}{\log 2}$$

$$dt = \frac{\pi}{\log 2} \cdot \frac{1}{x}\,dx$$

x	$1 \to 512$
t	$0 \to 9\pi$

$$= \int_0^{9\pi} \frac{\log 2}{\pi}\sin t\,dt$$

$$= \frac{\log 2}{\pi}\left[-\cos t\right]_0^{9\pi}$$

$$= \frac{\log 2}{\pi}\left\{1 - (-1)\right\}$$

$$= \underline{\frac{2\log 2}{\pi}}$$

$\boxed{1}$ $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos x + \sin x}{1 + \cos x \sin x}\, dx$ 【1997 同志社大】

$\sin x - \cos x = t$ とおく．

$(\cos x + \sin x)\,dx = dt$

$t^2 = 1 - 2\sin x \cos x$

x	$0 \to \frac{\pi}{6}$
t	$-1 \to \frac{1-\sqrt{3}}{2}$

$$= \int_{-1}^{\frac{1-\sqrt{3}}{2}} \frac{1}{1 + \frac{1-t^2}{2}}\, dt = \int_{-1}^{\frac{1-\sqrt{3}}{2}} \frac{2}{3 - t^2}\, dt$$

$$= \int_{\frac{1-\sqrt{3}}{2}}^{-1} \frac{2}{(t+\sqrt{3})(t-\sqrt{3})}\, dt$$

$$= \int_{\frac{1-\sqrt{3}}{2}}^{-1} \left(\frac{-\frac{1}{\sqrt{3}}}{t+\sqrt{3}} + \frac{\frac{1}{\sqrt{3}}}{t-\sqrt{3}} \right) dt$$

$$= \frac{1}{\sqrt{3}} \left[\log \left| \frac{t-\sqrt{3}}{t+\sqrt{3}} \right| \right]_{\frac{1-\sqrt{3}}{2}}^{-1}$$

$$= \frac{1}{\sqrt{3}} \log \frac{13+7\sqrt{3}}{22} .$$

$\boxed{2}$ $\displaystyle\int_0^1 \frac{dx}{(3+x^2)^2}$ 【2009 弘前大】

$x = \sqrt{3}\tan\theta \quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \qquad dx = \frac{\sqrt{3}}{\cos^2\theta}\,d\theta$

$x^2 + 3 = 3(\tan^2\theta + 1)$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{9(1+\tan^2\theta)^2} \cdot \frac{\sqrt{3}}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}}{9} \cdot \frac{1}{1+\tan^2\theta}\, d\theta$$

$$= \frac{\sqrt{3}}{9} \int_0^{\frac{\pi}{6}} \cos^2\theta\, d\theta = \frac{\sqrt{3}}{9} \int_0^{\frac{\pi}{6}} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{\sqrt{3}}{18} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}}{18} \left(\frac{\pi}{6} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right)$$

$$= \frac{\sqrt{3}}{18} \cdot \frac{2\pi + 3\sqrt{3}}{12}$$

$$= \frac{9 + 2\sqrt{3}\pi}{216} .$$

$\boxed{3}$ $\displaystyle\int_1^e \underbrace{(1+x)e^x}_{t} \underbrace{\log x}_{t'}\, dx$ 【1988 明治大】

$$= \left[x e^x \log x \right]_1^e - \int_1^e x e^x \cdot \frac{1}{x}\, dx$$

$$= \left[x e^x \log x - e^x \right]_1^e$$

$$= e \cdot e^e - e^e - (-e)$$

$$= e \cdot e^e - e^e + e .$$

$\boxed{4}$ $\displaystyle\int_1^{e^2} \frac{\sqrt{\log x}}{x}\, dx$ 【1993 法政大】

$\log x = t .$

$\frac{1}{x}\,dx = dt .$

$$= \int_0^2 \sqrt{t}\, dt$$

$$= \left[\frac{2}{3} t^{\frac{3}{2}} \right]_0^2$$

$$= \frac{2}{3} \cdot 2^{\frac{3}{2}}$$

$$= \frac{4}{3}\sqrt{2} .$$

$\boxed{1}$ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{6\sin x\cos x}{7\sin^2 x + 5\cos^2 x}\,dx$ 【1992 青山学院大】

$$= \int_0^{\frac{\pi}{2}} \frac{6\sin x\cos x}{5 + 2\sin^2 x}\,dx$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} \frac{4\sin x\cos x}{5 + 2\sin^2 x}\,dx$$

$$= \frac{3}{2}\Big[\log(5 + 2\sin^2 x)\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2}\left(\log 7 - \log 5\right)$$

$$= \underline{\frac{3}{2}\log\frac{7}{5}}.$$

$\boxed{2}$ $\displaystyle\int_0^2 x^3\sqrt{4-x^2}\,dx$ 【1993 横浜国立大】

$$\left(\begin{aligned} \sqrt{4-x^2} &= t \\ 4 - x^2 &= t^2 \\ -2x\,dx &= 2t\,dt \\ x\,dx &= (-t)\,dt \end{aligned}\right.$$

x	$0 \to 2$
t	$2 \to 0$

$x^2 = 4 - t^2$

$$= \int_2^0 (4 - t^2)\,t\cdot(-t)\,dt$$

$$= \int_2^0 (t^4 - 4t^2)\,dt$$

$$= \left[\frac{1}{5}t^5 - \frac{4}{3}t^3\right]_2^0$$

$$= 2^3\left(\frac{4}{3} - \frac{4}{5}\right)$$

$$= 8 \times \frac{8}{15} = \underline{\frac{64}{15}}.$$

$\boxed{3}$ $\displaystyle\int_0^{\pi} \underbrace{xe^x}_{t}\underbrace{\sin x}_{t'}\,dx = I$ 【1966 東京工業大】

$$I = \Big[(x-1)e^x\sin x - (x-2)e^x\cos x\Big]_0^{\pi} + \int_0^{\pi}(x-2)e^x(-\sin x)\,dx$$

$$= (\pi - 2)e^{\pi} - 2 - I + 2\int_0^{\pi} e^x\sin x\,dx$$

$$\therefore\ I = \frac{\pi-2}{2}e^{\pi} - 1 + \int_0^{\pi} e^x\sin x\,dx.$$

ここで、

$$\int_0^{\pi} e^x\sin x\,dx = \left[\frac{e^x(\sin x - \cos x)}{2}\right]_0^{\pi}$$

$$= \frac{e^{\pi}+1}{2}$$

より、$I = \dfrac{\pi-2}{2}e^{\pi} - 1 + \dfrac{e^{\pi}+1}{2} = \underline{\dfrac{(\pi-1)e^{\pi}-1}{2}}.$